Advanced Mechanics

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Exam, Tuesday, January 21, 2014, 9:00-12:00, room 5419.0119
4 problems (total of 60 points).
The solution of every problem on a separate piece of paper with name and student number.
Some useful formulas are listed at the end.

Problem 1 (15 pnts in total)

Consider the eom of a damped oscillator which is driven by a time-dependent force, $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t)/m$.

2 pnts a. Give the expression for the Greens function $G(t, t')$. Hint: look at the formulas at the end of this exam.

4 pnts b. Show, by direct substitution of $x(t) = G(t, t')$ that the Greens function solves the eom for $F(t) = 0$ at all times except $t = t'$.

4 pnts c. Give the explicit expression for $x(t)$ in terms of a definite integral for the case that

$$F(t) = \begin{cases} 0 & t < 0 \\ F_0 e^{-t/\tau} & t > 0 \end{cases} .$$

Distinguish two cases: $t < 0$, and $t > 0$. Do not solve integrals yet.

5 pnts d. Solve for $x(t)$ for $t > 0$ while using $\tau = 1/\beta$.

Problem 2 (15 pnts in total)

Given is a system of three masses, $m_1 = m$ at $\vec{r}_1 = (2b, 0, 0)$, $m_2 = m$ at $\vec{r}_2 = (0, 2b, 0)$, and $m_3 = 2m$ at $\vec{r}_3 = (-b\sqrt{2}, -b\sqrt{2}, 0)$, which are connected by rigid, massless rods.

2 pnts a. Determine the position of the center of mass of this system.

4 pnts b. Show that the inertial tensor for this system for rotations around the origin is

$$\{I\} = m\, b^2 \begin{pmatrix} 8 & -4 & 0 \\ -4 & 8 & 0 \\ 0 & 0 & 16 \end{pmatrix}$$

4 pnts c. Determine the principal axes and the corresponding momenta of inertia.

2 pnts d. The system is rotating with $\vec{\omega} = \omega_0(1, 1, 0)$ determine $\vec{L}$ and $\vec{N} = d\vec{L}/dt$.

3 pnts e. The system is rotating with $\vec{\omega} = \omega_0(1, 0, 0)$ determine $\vec{L}$ and $\vec{N} = d\vec{L}/dt$.

Problem 3 (10 pnts in total)

A bullet is shot straight up (along a plumb line) at a latitude λ (equator is at zero, North is positive) on Earth. The bullet reaches a height of 500 m and then falls back to Earth. Ignore friction in this problem and assume that the Earth rotates with a constant angular frequency $\omega_E = 7.3 \times 10^{-5} s^{-1}$. Take $g_0 = 10 m/s^2$, $R_{\text{Earth}} = 6.4 \times 10^6$ m and assume Earth to be a perfect sphere. Take as Earth coordinates: $\hat{x}$ =South ; $\hat{y}$ =East; $\hat{z}$ = vertical up in radial direction.

1 pnts a. Express the rotation vector of the Earth in the local coordinate system. Show that this expression has the right limits for the North pole, the Equator and the South pole.

2 pnts b. Calculate (in leading order in ω_E) the deviation of a plumb line from a normal to the Earth surface (= purely radial direction).

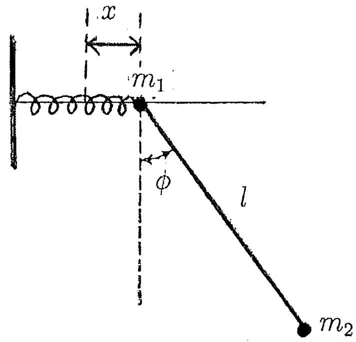
1 pnts c. Give the expression for velocity as function of height (ignore corrections due to Earth rotation and friction).

4 pnts d. Calculate (in leading order in ω_E) the displacement (may round to whole numbers) from the vertical when the bullet has reached the highest point when this experiment is performed on the Equator. Give the direction.

2 pnts e. Same as the previous, but now on the North pole.

Problem 4 (20 pnts in total)

Consider the system shown in the diagram. A mass m_1 is free to move along a horizontal line and is connected to a fixed point by an ideal, massless spring with spring constant k. One end of a massless, inextensible rod of length l is free to pivot about m_1. A point mass m_2 is fixed to the other end of the rod. The motions of masses m_1 and m_2 are confined to the same vertical plane and take $m_1 = m_2 = m$.



3 pnts a. Derive the expression for the kinetic energy of m_2.

2 pnts b. Show that the Lagrangian can be written in terms of the variables in the figure as

$$L = m\dot{x}^2 + ml^2\dot{\phi}^2/2 + ml\cos\phi\,\dot{x}\,\dot{\phi} - kx^2/2 + mgl\cos\phi$$

2 pnts c. Give the equations of motion.

2 pnts d. Show that in the limit of small oscillations the equations of motion reduce to
$l\ddot{\phi} + \ddot{x} = -g\phi$; $\quad ml\ddot{\phi} + 2m\ddot{x} = -kx$.
Justify carefully each approximation you make.

3 pnts e.Show that the angular frequencies, ω, of small amplitude oscillations satisfy the equation

$$lm\omega^4 - \omega^2(2gm + kl) + gk = 0$$

2 pnts f. It the equilibrium solution $x = 0$, $\phi = 0$ stable?

3 pnts g. Give the expression for the generalized momenta p_x and p_θ.

3 pnts h. Determine the Hamiltonian.

Possibly useful formulas:
$\vec{F}' = \vec{F}_{\text{inert}} - 2m\vec{\omega}\times\vec{v}' - m\dot{\vec{\omega}}\times\vec{r}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$, and $\vec{v} = \vec{v}' + \vec{\omega}\times\vec{r}'$

$$\int xe^{cx}dx = \frac{e^{cx}}{c^2}(cx-1)$$

The response of a damped oscillator $\ddot{x} + 2\beta\dot{x} + \omega_r^2 x = F(t)/m$ to a delta force at $t = 0$ is $\frac{1}{\omega_1 m}e^{-\beta t}\sin\omega_1 t$ for $t > 0$, where $\omega_1 = \sqrt{\omega_r^2 - \beta^2}$.
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$; $\quad \cos(\alpha-\beta) = \sin\alpha\sin\beta + \cos\alpha\cos\beta$
$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$

Integrals

For $c > 0$ we have:

$$\int e^{cx}dx = \frac{1}{c}e^{cx}\ ;\ \int x\,e^{cx}dx = \frac{cx-1}{c^2}e^{cx}\ ;\ \int x^2e^{cx}dx = \frac{c^2x^2-2cx+2}{c^3}e^{cx}$$